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New Approach to Perturbation Theory of Many-Particle Systems

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We derive an infinite hierarchy of integral equations for the Green functions of a many-particle system. This set of equations forms the basis of a unified approach to the perturbation theory of many boson and many fermion systems and avoids the introduction of the adiabatic hypothesis. It is demonstrated how a well-known ground state perturbation theory of a system of interacting fermions is obtained without introducing disconnected diagrams. It is shown that the formalism allows a self-consistent determination of the condensate Green function of a condensed Bose system and a derivation of the Beliaev, Hugenholtz, and Pines result for the single-particle $\mathbf{k} \neq 0$ Green function is given. A new self-consistent equation for the $\mathbf{k} = 0$ Green function is solved to yield the well-known self-energy relation $\Sigma_{11} - \Sigma_{02} = \mu$ which plays the role of a self-consistency condition on the theory.

1. INTRODUCTION

An exact, first-principles, many-body perturbation theory description of the properties of liquid ^{4}He is still lacking. To examine the possibility of condensation into the $\mathbf{k} = 0$ mode, it is natural to focus attention on the single-particle condensate (i.e., $\mathbf{k} = 0$) Green function $g_0(t - t')$. Denoting the normalized interacting ground state by $|\Psi_N\rangle$ and a Heisenberg picture annihilation operator for the $\mathbf{k} = 0$ mode by $a_{0_H}(t)$, this quantity is defined (Beliaev, 1958)

$$iG_0(t - t') = \langle \Psi_N | T\left[a_{0_H}(t) a_{0_H}^{\dagger}(t')\right] | \Psi_N \rangle \tag{1}$$

For a condensed system and employing the Heisenberg picture of Hugenholtz and Pines (1959) (Beliaev, 1958), one expects on the basis of

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physical arguments that

$$iG_0(t-t') = n_0 \tag{2a}$$

$$\left(iG_0(t-t') = n_0 e^{-i\mu(t-t')}\right) \tag{2b}$$

Here n_0 is the particle density in the condensate and μ the chemical potential. Ideally, one would like to start from a purely microscopic theory capable of describing a Bose system which may or may not possess a condensate and determine under what conditions equations (2) are valid.

In the approach of Hugenholtz and Pines (1959) the validity of equation (2a) is guaranteed from the outset by replacing the $\mathbf{k} = 0$ operators by the c number n_0. Thus, the theory is restricted to the descriptions of a condensed Bose system only.

An alternative formalism, which might be more readily adapted to describe a noncondensed system, has been provided by Beliaev (1958). In principle this approach might be adopted to evaluate G_0, without introducing assumptions concerning the $\mathbf{k} = 0$ mode. However, it is shown in Section 2 that this formalism leads to inconsistencies when one attempts to insert equations of the form of (2) into the diagrammatic perturbation series which result from application of the adiabatic hypothesis. It would thus appear that adiabatic switching must be avoided if equations of the form of (2) are to be employed self-consistently in the analysis.

Most modern formulations of perturbation theory avoid adiabatic switching (Amit, 1978). Unfortunately they do not readily lend themselves to a Beliaev type of approach. For this reason, in the present paper, we introduce in Section 3 a new approach to perturbation theory. The resulting formalism is applicable to both many-fermion and many-boson systems and allows an approach to the many-boson problem which is free from the inconsistencies discussed above. Furthermore it yields an exact ground state many-boson perturbation theory which possesses a manageable diagrammatic structure. The theory is applied to a noncondensed system in Section 4 and to a condensed system in Section 5.

2. CONVENTIONAL PERTURBATION THEORY AND ADIABATIC SWITCHING

In the usual approach to perturbation theory one introduces the notion of adiabatic switching. To this end, the time-independent Hamiltonian $H = H_0 + H_1$ is replaced by

$$H^\epsilon = H_0 + e^{-\epsilon|t|}H_1$$

Employing the well-known Gell-Mann and Low (1950) procedure the

interacting ground state $|\Psi_N\rangle$ can be generated from the noninteracting ground state $|\Phi_N\rangle$ in the usual manner (Fetter and Walecka, 1971). For our purposes we simply note that we can write, in symbolic notation,

$$|\Psi_N\rangle = \lim_{\varepsilon = 0^+} \left[\Psi_N^\varepsilon\right]$$

Similarly the single-particle Green function $G(1;1')$ defined by (Fetter and Walecka, 1971)

$$G(1;1') = -i\langle\Psi_N|T\left[\psi_H(1)\psi_H^\dagger(1')\right]|\Psi_N\rangle$$

can also be written in the symbolic form

$$iG(1;1') = \left[\lim_{\varepsilon = 0^+} \langle\Psi_N^\varepsilon|\right]\left[\lim_{\varepsilon = 0^+} T\{\psi_{H^\varepsilon}(1)\psi_{H^\varepsilon}^\dagger(1')\}\right]\left[\lim_{\varepsilon = 0^+} |\Psi_N^\varepsilon\rangle\right]$$

Within conventional perturbation theory, one assumes that the product of the limits equals the limit of the product and hence we obtain

$$iG(1;1') = i \lim_{\varepsilon = 0^+} \left[G_\varepsilon(1;1')\right]$$

where we have defined

$$iG_\varepsilon(1;1') = \langle\Psi_N^\varepsilon|T\{\psi_{H^\varepsilon}(1)\psi_{H^\varepsilon}^\dagger(1')\}|\Psi_N^\varepsilon\rangle$$

With this as a prelude we now consider evaluating the condensate Green function. To this end we take a system of N interacting bosons of mass m enclosed in a volume V and suppose N and V to become infinite while the density $n = N/V$ remains finite. If the ground state of the noninteracting system is denoted by $|\Phi_N\rangle$ where

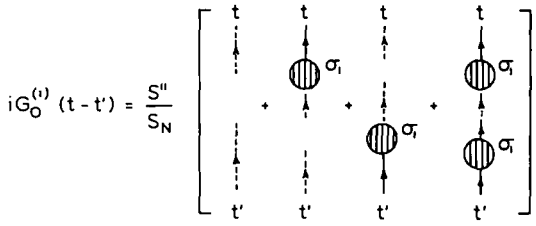
$$|\Phi_N\rangle = (N!)^{-1/2}\left(a_0^\dagger\right)^N|0\rangle$$

the condensate Green's function in the interaction picture is, in an obvious notation,

$$iG_0^{(1)}(t-t') = \frac{\langle\Phi_N|T\left[\alpha(t)\alpha^\dagger(t')S\right]|\Phi_N\rangle}{\langle\Phi_N|S|\Phi_N\rangle} \tag{3}$$

where S is the S-matrix operator. Employing standard procedures (Beliaev, 1958) we may readily expand $iG_0^{(1)}(t-t')$ in the form shown in equation

(4):



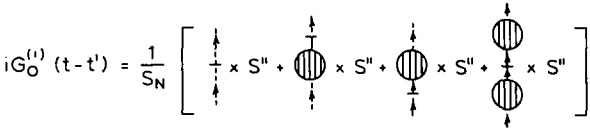
(4)

where a dotted line entering (leaving) the point $x = \mathbf{x}t_x$ represents an operator $\alpha(t_x)[\alpha^\dagger(t_x)]$ and a solid line running from y to x represents a factor $g(x, y)$, where

$$g(x, y) = \langle 0|T[\psi(x)\psi^\dagger(y)]|0\rangle \tag{5}$$

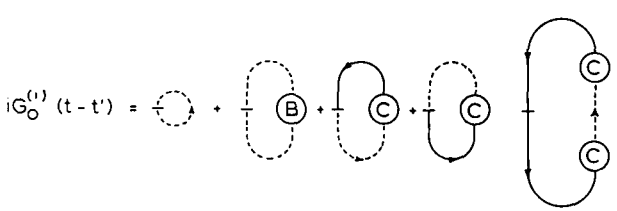
(It is to be emphasized that the self-energy σ_1 is defined by this equation.)

Although S'' is topologically identical with S_N, the two do not cancel. However there is a well-known method of overcoming this problem (Brandow, 1971). In order to apply this method we first rewrite equation (4) in the following form:



(6)

The procedure then is simply to connect the dotted lines to each other in all possible ways to form continuous lines and impose the rule "associate a condensate weight $(-1)^{l+c}V^{-c}N^l$ with every diagram containing l loops and c condensate lines." Cancellation of disconnected diagrams by S_N in equation (6) now occurs, but at the expense of greatly increasing the number of connected diagrams. After changing dotted lines into condensate lines and allowing for cancellation of disconnected diagrams equation (6) takes the form

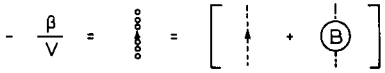


(7)

which defines the self-energies $\Sigma^B = \textcircled{B}$ and $\Sigma^C = \textcircled{C}$.

As a result of the large number of connected diagrams generated by the above procedure, the resultant complexity makes the neglect of diagrams of relative order V^{-1} highly desirable. It can readily be seen that the leading diagrams in the perturbation series expansion are bubble diagrams, all other diagrams being of relative order V^{-1}. Because of this these diagrams are the only ones ever considered by many authors (see, for example, Brandow, 1971, and Lee, 1971). We therefore consider the summation of this subset of terms on the right-hand side of equation (7).

In order to carry out the summation of all the bubble diagrams consider first the quantity β defined by the equation

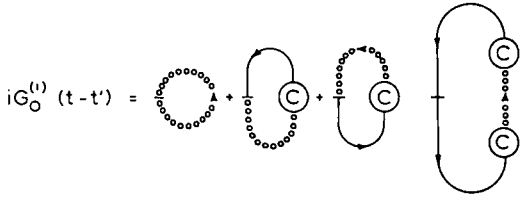


(8)

With this definition it is clear that the sum of the first two terms on the right-hand side of equation (7) is simply $n\beta$ which we denote as follows:

(9)

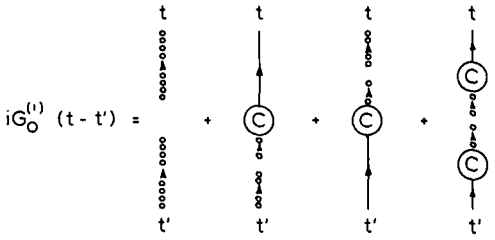
Now β is obtained from the first term on the right-hand side of equation (6) by allowing the dotted lines connected to the horizontal bar to pair with dotted lines is S''. It is clear that this procedure can be carried out on all the other terms in this same equation. We may therefore omit all contributions to Σ^C that contain "disjointed" parts and reinterpret a condensate line as $(-\beta/V)$ rather than simply $(-1/V)$. (A disjointed part is that part of a self-energy insertion which becomes disconnected from the rest of the diagram when all the condensate lines are removed.) It now readily follows that we can rewrite equation (7) in the form



(10)

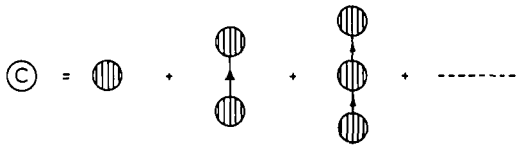
Since, for the present subset of diagrams, a diagram containing m renormalized condensate lines also contains m loops, we may change the rule for obtaining condensate weights to associate a factor $(\beta n)^m$ with any diagram containing m condensate lines. In view of this we can rewrite equation (10)

in the form



(11)

The self-energy Σ^C may now be expanded in terms of a proper self-energy Σ^{*C} denoted by ◍ as shown in equation (12):



(12)

Employing the usual definition of a Fourier transform (Fetter and Walecka, 1971) we obtain from equation (12) the result

$$\Sigma^C_{(k)} = \Sigma^{*C}_{(k)}\left[1 - g(k)\Sigma^{*C}_{(k)}\right]^{-1}$$

where [see equation (5)]

$$g(k) = \left[\omega - \omega_k + i\eta\right]^{-1}$$

Substituting these two results into equation (11) gives

$$iG_0^{(1)}(t - t') = \beta n\left[1 - g(0)\Sigma^{*C}_{(0)}\right]^{-2} \tag{13}$$

The two sides of this equation are mathematically inconsistent since the right-hand side is time independent, whereas the left-hand side is a function of $(t - t')$. It must be emphasized that this noncompatibility of the time dependencies is *not* simply a consequence of our defining the interaction picture in terms of H_0 rather than $K_0 = H_0 - \mu N$. Although $\alpha(t) = \exp(iH_0 t/\hbar)V^{-1/2}a_0\exp(-iH_0 t/\hbar)$ is time independent, the right-hand side of equation (3) is time dependent; the dependence on t and t' arising from the presence of the Wick time-ordering operator.

The above analysis could have been carried out in the interaction picture of the Hamiltonian $K_0 = H_0 - \mu N$ thus giving the operator $\alpha(t)$ an explicit time dependence $\exp(i\mu t/\hbar)$. This would however lead us to define our Heisenberg picture in terms of the Hamiltonian $H_0 + H_1 - \mu N$ and in this picture equation (13) is simply multiplied on both sides by a factor $\exp[i\mu(t - t')/\hbar]$, which clearly will not affect the argument given above.

Hence, in summary, we see that employing conventional perturbation theory leads to a mathematical inconsistency which we attribute to the adiabatic switching device. In the following sections we therefore consider a new approach to perturbation theory which avoids the concept of adiabatic switching as it is normally applied in the definitions of the ground state wave function and the Green functions of a system.

3. A HIERARCHY OF EQUATIONS FOR THE GREEN FUNCTIONS OF A MANY-PARTICLE SYSTEM

Our aim is to introduce a new hierarchy of integral equations for the various Green functions of a many-particle system. This set of equations is analogous to the system of coupled differential equations which arises within the equation-of-motion method. However, the hierarchy of equations presented below has the advantage that it is readily analyzed by diagrammatic techniques. This feature provides us with a particularly simple and transparent approach to perturbation theory, which is discussed in detail in Section 4.

The Green functions of a many-particle system may be defined by

$$(i)^{(m+n)/2} G^{(m,n)}_{(1\ldots m;1'\ldots n')} = \langle \Psi_N | T\left[\psi_H(1)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(n')\right] | \Psi_N \rangle \tag{14}$$

In this equation, $m+n$ is taken to be even and the Heisenberg picture has been employed.

To analyze the right-hand side of equation (14), we first prove a theorem on time-ordered products of Heisenberg picture field operators. To this end, we note that as a consequence of the equation of motion of the time development operator (Fetter and Walecka, 1971) a time-ordered product of creation and annihilation operators satisfies the following relation, which is derived in Appendix A:

$$\begin{aligned}
&e^{-\varepsilon|t_1|}e^{i\omega_k t_1}T\{a_{H\mathbf{k}}(t_1)\psi_H(2)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(n')\} \\
&\quad - e^{-\varepsilon|t'|}e^{i\omega_k t'}T\{a_{H\mathbf{k}}(t')\psi_H(2)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(n')\} \\
&\quad = \sum_{j=1}^{n} S^{m+j}T\{\psi_H(2)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(j'-1)[a_{H\mathbf{k}}(t_j'),\psi_H^\dagger(j')]_{-S} \\
&\quad \times \psi_H^\dagger(j'+1)\ldots\psi_H^\dagger(n')\}\{\theta(t_1-t_j')-\theta(t'-t_j')\}e^{-\varepsilon|t_j'|}e^{i\omega_k t_j'} \\
&\quad - i\int_{t'}^{t_1} dt_p T\{X_H(t_p)\psi_H(2)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(n')\}e^{i\omega_k t_p}
\end{aligned} \tag{15}$$

where

$$X_H(t_p) = \left(e^{iHt_p}[a_{\mathbf{k}}, H_1]e^{-iHt_p} - i\varepsilon \operatorname{sgn}(t_p) a_{H\mathbf{k}}(t_p)\right) e^{-\varepsilon|t_p|} \tag{16}$$

Note that equation (15) involves an (anti-) commutator of operators at equal times. This is to be regarded as a single entity which commutes with all quantities under the time-ordering operator. Alternatively, one may write

$$\left[a_{H\mathbf{k}}(t_j'), \psi_H^\dagger(j')\right]_{-S} = \lim_{\eta \to 0^+} \left\{a_{H\mathbf{k}}(t_j' + \eta)\psi_H^\dagger(j') - S\psi_H^\dagger(j') a_{H\mathbf{k}}(t_j' - \eta)\right\} \tag{17}$$

which avoids the ambiguities which arise when T operates on quantities at equal times.

In the limit $t' = \pm\infty$, the second term on the left-hand side of equation (15) vanishes. As will become clear later, the choice $t' = +\infty$ is analogous to the transformation from particles to holes in the theory of fermions. For this reason, it is convenient to introduce the function $F(\mathbf{k})$, which can take on values $+1$ or 0 depending on the value of the argument $\mathbf{k}$. For values of $\mathbf{k}$ such that $F(\mathbf{k}) = 1(0)$ the limit $t' = -\infty(+\infty)$ is taken in equation (15). This leads to the result

$$\begin{aligned} e^{-\varepsilon|t_1|} T\{a_{H\mathbf{k}}(t_1)\psi_H(2)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(n')\} \\ = \sum_{j=1}^{n} S^{m+j} T\{\psi_H(2)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(j'-1) \\ \times \left[a_{H\mathbf{k}}(t_j), \psi_H^\dagger(j')\right]_{-S} \psi_H^\dagger(j'+1)\ldots\psi_H^\dagger(n')\} i g_{\mathbf{k}}(t_1 - t_j') e^{-\varepsilon|t_j'|} \\ + \int_{-\infty}^{\infty} dt_p\, T\{X_H(t_p)\psi_H(2)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(n')\} g_{\mathbf{k}}(t_1 - t_p) \end{aligned} \tag{18}$$

where we have defined

$$i g_{\mathbf{k}}(t_1 - t_j') = e^{-iw_{\mathbf{k}}(t_1 - t_j')}\left[F(\mathbf{k})\theta(t_1 - t_j') - (1 - F(\mathbf{k}))\theta(t_j' - t_1)\right] \tag{19}$$

It follows from the definitions of the various operators that

$$\left[a_{H\mathbf{k}}(t_j'), \psi_H^\dagger(j')\right]_{-S} = V^{-1/2} e^{-i\mathbf{k}\cdot\mathbf{x}_j'} \tag{20}$$

and

$$\begin{aligned} X_H(t_p) = V^{-1/2} \int dt_p' \int d^3X_p \int d^3X_p' U(p - p') e^{-i\mathbf{k}\cdot\mathbf{x}_p} \psi_H^\dagger(p')\psi_H(p')\psi_H(p) \\ - i\varepsilon \operatorname{sgn}(t_p) a_{H\mathbf{k}}(t_p) e^{-\varepsilon|t_p|} \end{aligned} \tag{21}$$

where we have written

$$U^{\varepsilon}(p-p') \equiv U(\mathbf{x}_p - \mathbf{x}'_p)\,\delta(t_p - t'_p)\,e^{-\varepsilon|t_p|} \tag{22}$$

Hence after substituting equations (19) and (20) into (18) we obtain

$$\begin{aligned} e^{-\varepsilon|t_1|}T\{\psi_H(1)\psi_H(2)\ldots\psi_H(m)\psi_H^{\dagger}(1')\ldots\psi_H^{\dagger}(n')\} \\ = \sum_{j=1}^{n} S^{m+j} ig(1-j')\,e^{-\varepsilon|t'_j|}T\{\psi_H(2)\ldots\psi_H(m)\psi_H^{\dagger}(1')\ldots\psi_H^{\dagger}(j'-1) \\ \times\psi_H^{\dagger}(j'+1)\ldots\psi_H^{\dagger}(n')\} \\ + \int dP\,dP'\,U^{\varepsilon}(p-p')\,g(1-p')\,T\{\psi_H^{\dagger}(p'_+)\psi_H(p')\psi_H(p_-)\psi_H(2) \\ \times\ldots\psi_H(m)\psi_H^{\dagger}(1')\ldots\psi_H^{\dagger}(n')\} - iV^{-1/2}\sum_{\mathbf{k}=0}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{x}_1} I_{\mathbf{k}}^{(\varepsilon)}(t_1) \end{aligned} \tag{23}$$

where

$$\begin{aligned} I_{\mathbf{k}}^{(\varepsilon)}(t_1) = \varepsilon\int_{-\infty}^{\infty} dt_p\, g_{\mathbf{k}}(t_1 - t_p)\,\mathrm{sgn}(t_p)\,e^{-\varepsilon|t_p|} \\ \times T\{a_{H\mathbf{k}}(t_p)\psi_H(2)\ldots\psi_H(m)\psi_H^{\dagger}(1')\ldots\psi_H^{\dagger}(n')\} \end{aligned} \tag{24}$$

$$g(j-j') = \sum_{\mathbf{k}=0}^{\infty} V^{-1} e^{i\mathbf{k}\cdot(\mathbf{x}_j - \mathbf{x}'_j)} g_{\mathbf{k}}(t_j - t'_j) \tag{25}$$

and we have employed the notation

$$\int dP\,dP' \equiv \int_{-\infty}^{\infty} dt_p\,dt'_p \int_V d^3x_p\,d^3x'_p \tag{26}$$

Note that the subscripts + and − in the second term on the right-hand side of equation (23) indicate that a suitable interpretation [cf. equation (17)] of the arguments of operators at equal times is to be employed.

Since $I_{\mathbf{k}}^{(\varepsilon)}(t_1)$ vanishes if the correct limiting procedure is adopted, we are now in a position to state the theorem which leads to the desired

hierarchy of equations:

$$T\{\psi_H(1)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(n')\}$$

$$= \sum_{j=1}^{n} S^{m+j} i g(1-j') T\{\psi_H(2)\ldots\psi_H(m)\psi_H^\dagger(1')$$

$$\times\ldots\psi_H^\dagger(j'-1)\psi_H^\dagger(j'+1)\ldots\psi_H^\dagger(n')\}$$

$$+ \lim_{\varepsilon=0^+} \int dp\, dp' U^\varepsilon(p-p') g(1-p) T$$

$$\times\{\psi_H^\dagger(p'_+)\psi_H(p')\psi_H(p_-)\psi_H(2)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(n')\} \tag{27}$$

Consider a many-particle system to which the above theorem may be applied. A set of equations for the Green function is obtained after taking matrix elements with respect to the ground state $|\Psi_N\rangle$. From the definition (14), one obtains

$$G^{(m,n)}_{(1\ldots m;\,1'\ldots n')} = \sum_{j=1}^{n} S^{m+j} g(1-j') G^{(m-1,n-1)}_{(2\ldots m;\,1'\ldots\ j'-1,\,j'+1\ldots n')}$$

$$+ i\int dP\, dP' U^\varepsilon(p-p') g(1-p) G^{(m+1,n+1)}_{(2\ldots m,\,p_-,\,p'_+;\,1'\ldots n',\,p'_+)} \tag{28a}$$

where the limit $\varepsilon = 0^+$ is implied.

A "conjugate" set of equations is obtained by taking the Hermitian conjugate of the equation obtained from (28a) after replacing all times by their negative and H by $-H$. The result is

$$G^{(n,m)}_{(n'\ldots 1';\,m\ldots 1)} = \sum_{j=1}^{n} S^{m+j} g(j'-1) G^{(n-1,m-1)}_{(n'\ldots\ j'+1,\,j'-1,\,\ldots 1';\,m\ldots 2)}$$

$$+ i\int dP\, dP' U^\varepsilon(p-p') g(p-1) G^{(n+1,m+1)}_{(p'_-,\,n',\,\ldots 1';\,p',\,p_+,\,m\ldots 2)} \tag{28b}$$

Equations (28a) and (28b) define the hierarchy of equations which we have been seeking and form the basis of our approach to perturbation theory.

4. GROUND STATE PERTURBATION THEORY OF NONCONDENSED SYSTEMS

To illustrate how equations (28) may be employed to yield a perturbation theory, we shall proceed to evaluate the single-particle Green function $G^{(1,1)}$. To this end we introduce the following rules which allow equations (28) to be written in terms of diagrams.

(i) Denote $G^{(m,n)}_{(1\ldots m;\,1'\ldots n')}$ by $m(n)$ thick solid lines. One end of each line is to be left free and the other end is to enter (leave) one of the points $1\ldots m$ $(1'\ldots n')$.

(ii) The lines which correspond to the unprimed (primed) labels of $G^{(m,n)}_{(1\ldots m;\,1'\ldots n')}$ are to be arranged in the order in which they appear in the argument. The line corresponding to 1 (1′) is to be placed furthest to the left and that corresponding to $m(n')$ is to be placed furthest to the right.

These rules identify two sets of lines. One set refers to labels which appear after the semicolon in the argument of $G^{(m,n)}$ and the other set refers to those which appear before. Although the relative order of lines within a given set is significant, the position of the lines of one set relative to those of the other is not. However, when convenient, we shall place the lines which refer to labels appearing before the semicolon of $G^{(m,n)}$ above those which refer to labels appearing after. In this way, rules (i) and (ii) lead to the representation of the left-hand side of equation (28a) shown in Figure 1.

Diagrammatic analysis of the right-hand side of equations (28) requires the introduction of further rules:

(iii) Denote $g(j, j')$ by a thin solid line running from j' to j.

(iv) Denote the potential $iU^{\varepsilon}(j - j')$ by a wavy line joining the points j and j'.

(v) Integrate over all internal coordinates [i.e., P and P' of equations (28)].

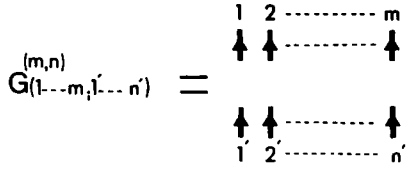


Fig. 1

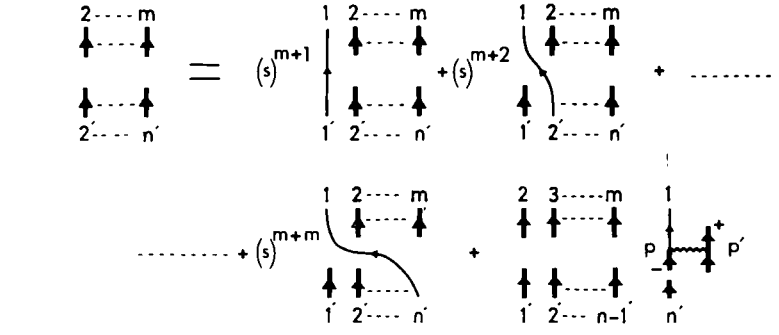
Employing these rules, equations (28) take the form of equations (29) of Figure 2 where the quantity S equals $+1$ for bosons and -1 for fermions.

As a special case of rule (i), it is convenient to introduce the following:

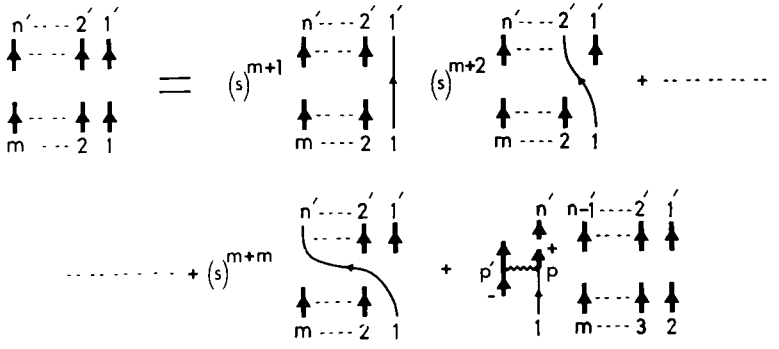
(ib) Denote $G^{(1,1)}_{(i;i')} \equiv G(i;i')$ by a thick solid line running from i' to i.

As examples of the use of rule (ib), Figure 3 shows the $m = n = 1$ and $m = n = 2$ versions of equations (28a) and (28b), respectively. For convenience the dummy internal coordinates will no longer be shown explicitly.

It is now easy to see how diagrammatic perturbation theory arises from the hierarchy of equations. To this end substitute equation (31) for the two-particle Green function on the right-hand side of equation (30). The result is shown in Figure 4 where the last line follows from the fact that operators (anti-) commute under the T operator. The last term on the right-hand side of equation (32) is evaluated by substituting equation (29b)

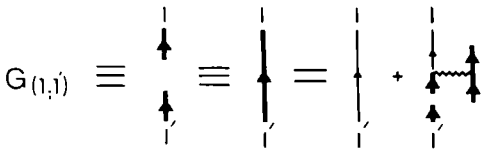


(29a)

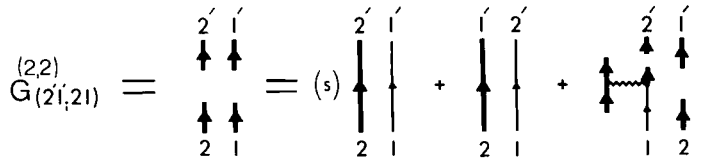


(29b)

Fig. 2

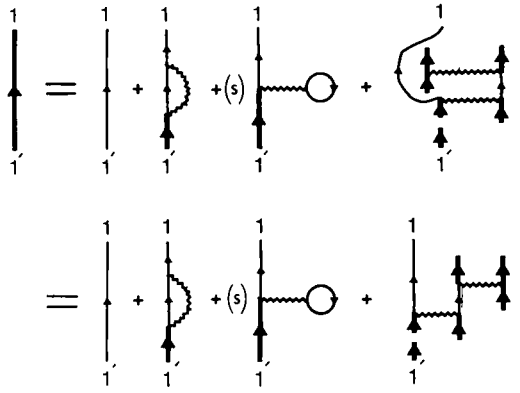


(30)



(31)

Fig. 3



(32)

Fig. 4

for the three-particle Green function. The result contains terms involving $G^{(2,2)}$ and $G^{(4,4)}$ which may be further expanded by employing equations (31) and (29b). Continuing this iterative process to infinity shows that the hierarchy of equations may be iterated self-consistently to yield a perturbation series, which is topologically identical with the usual fermion series obtained by employing the adiabatic hypothesis.

In view of this, we note (Fetter and Walecka, 1971) that the factors S may be omitted provided one introduces the following rule:

(vi) Associate a factor S^L with each diagram containing L closed loops.

The present approach to perturbation theory possesses many advantages over the conventional method. Disconnected diagrams are avoided

and Dyson's equations appear naturally at each order of perturbation theory [cf. the first two terms on the right-hand side of equation (32)]. Notice that the last term on the right-hand side of equation (32) contains a three-particle Green function $G^{(3,3)}$. As further iterations are carried out, terms involving higher-order Green functions $G^{(m,m)}$ appear. On physical grounds, one expects that for a system which does not exhibit long-range order (L.R.O.),

$$\lim_{m \to \infty} G^{(m,m)} = 0 \tag{33}$$

and hence the series obtained from iterative procedure should converge and be immediately applicable as it stands. However, in the presence of L.R.O., one expects equation (33) to be violated and less direct methods are required. An example of this situation is given in the following section.

The function $F(\mathbf{k})$ has yet to be specified. Within the conventional approach, $F(\mathbf{k})$ is fixed by invoking the adiabatic hypothesis. Essentially this consists of assuming that in the limit $H_0 \to H$, $|\Phi_N\rangle$ becomes identical with $|\Psi_N\rangle$. Within the present formalism, a similar technique can be applied. Considering first the situation arising when $S = -1$ (corresponding to fermions), we note that *if it is assumed* that in the limit $H \to H_0$, $|\Psi_N\rangle$ becomes identical with $|\Phi_N\rangle$, then the choice $F(\mathbf{k}) = \theta(\mathbf{k} - \mathbf{k}_F)$ must be made. In this situation, the fermion perturbation theory obtained from the present hierarchy of equations is identical with the perturbation theory obtained by the conventional approach.

For bosons ($S = +1$) the assumption that $H \to H_0$ implies $|\Psi_N\rangle \to |\Phi_N\rangle$ leads to the choice $F(\mathbf{k}) = 1 - \delta_{\mathbf{k},\mathbf{0}}$. In addition since H commutes with the number operator, one concludes that (in the thermodynamic limit) the perturbation series generated from equation (32) describes a zero density Bose gas only. Thus one is led to expect that for a *finite density* Bose fluid in its ground state either the Green functions are nonanalytic, or the condition (33) is violated. For a condensed Bose system described by equations such as (2), the latter expectation is borne out and it is of interest to consider the perturbation theory which arises from the hierarchy of equations in this situation. We address ourselves to this application of the present formalism in the following section. However, before proceeding, it is convenient to end this section with a further remark concerning the function $F(\mathbf{k})$.

In general, the validity of the limiting procedures discussed above are questionable when the exact Green functions of the interacting systems are nonanalytic in the region $H \to H_0$. Within the present formalism, the function $F(\mathbf{k})$ is not fixed from the outset and one may calculate such

physical quantities as the ground state energy E as a functional of $F(\mathbf{k})$. In principle, the function $F(\mathbf{k})$ may then be obtained by minimizing E with respect to F. Thus, we are led to determine F from the equation

$$\delta E[F(\mathbf{k})] = 0 \tag{34}$$

which is expected to have important applications in situations in which the above limiting procedures fail.

5. GROUND STATE PERTURBATION THEORY OF A CONDENSED BOSE SYSTEM

In the previous section, we demonstrated that the present approach may be successfully employed to yield a perturbation theory of a finite density Fermi system and a zero density Bose "gas." In the present section we show how the hierarchy of equations may be solved iteratively to yield a perturbation theory of a finite density Bose system. It is demonstrated that inconsistencies no longer arise from the $\mathbf{k} = 0$ contribution. Instead, there appears a self-consistent equation for the $\mathbf{k} = 0$ Green function, which leads to a well-known relation among the self-energies of the theory. The final result is equivalent to the well-known Hugenholtz and Pines theory.

In what follows, we follow Beliaev (1958) and employ the Heisenberg picture of the Hamiltonian H and not the Heisenberg picture of the Hamiltonian $H - \mu\hat{N}$. This is permissible, because in the present formalism H commutes with the number operator $\hat{N}$. In addition, we set

$$F(\mathbf{k}) = 1 - \delta_{\mathbf{k},\mathbf{0}}$$

In view of the fact that equation (33) is violated by the condensate Green functions, it is convenient to separate the contributions from the $\mathbf{k} = 0$ mode by writing

$$\psi(x) = \psi'(x) + V^{-1/2}a_0$$

where

$$\psi'(x) = V^{-1/2} \sum_{\mathbf{k} \neq 0} e^{-i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} \tag{35}$$

By analogy with equation (14), the noncondensate Green functions $G'^{(m,n)}$

are defined by

$$(i)^{(m+n)/2} G'^{(m,n)}_{(1\ldots m;\,1'\ldots n')} = \langle \Psi_N | T\left[\psi'_H(1)\ldots\psi'_H(m)\psi'^{\dagger}_H(1')\ldots\psi'^{\dagger}_H(n')\right] | \Psi_N \rangle$$

With this definition, the Green functions of equation (14) may be written as a sum of products of their condensate and noncondensate components. As a simple example, we note that since H commutes with the momentum operator,

$$G^{(1,1)}(1;1') = G'^{(1,1)}(1;1') + G_0^{(1,1)}(1;1') \tag{36}$$

In order to emphasize this separation, it is convenient to modify the diagram rules (i) to (iii) of the previous section. Two new rules [referred to as (i') and (ii')] are obtained from rules (i) and (ii) by replacing $G^{(m,n)}$ by $G'^{(m,n)}$. In this way, thick solid lines now refer to noncondensate Green functions only.

Similarly, defining the quantity $g'(j-j')$ by [cf. equation (25)]

$$g'(j-j') = V^{-1} \sum_{\mathbf{k}\neq 0} e^{-i\mathbf{k}(\mathbf{x}_j - \mathbf{x}'_j)} g_{\mathbf{k}}(t_j - t'_j) \tag{37}$$

rule (iii) is now changed to the following:

(iii') Denote $g'(j, j')$ by a thin solid line running from j' to j.

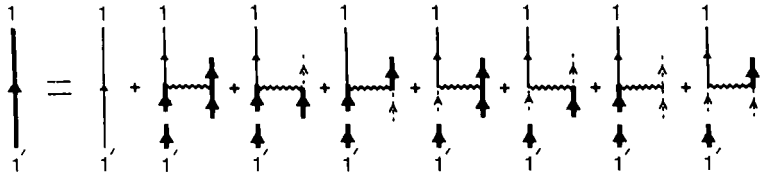
The $\mathbf{k} = 0$ contributions are described by the rules:

(vii) Denote $G_0^{(m,n)}(t_1\ldots t_m; t'_1\ldots t'_n)$ by m (n) dashed lines. One end of each line is left free and the other is to enter (leave) one of the points $t_1\ldots t_m(t'_1\ldots t'_n)$.

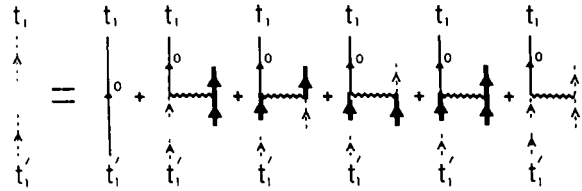
(viii) Denote $V^{-1}g_0(t_j, t'_j)$ by a thin solid line running from t_j to t'_j and associate a "0" with the arrow of the line (cf. the first term on the right-hand side of equation (39)).

Employing the above rules and rules (iv) and (v) of the previous section, equation (30) now becomes the two separate equations shown in Figure 5.

In order to make contact with the theory of Hugenholtz and Pines, we proceed to evaluate the single-particle Green functions. To this end, equations (29) are written in terms of their condensate and noncondensate components and the noncondensate Green functions appearing on the right-hand side of equations (38) and (39) are evaluated self-consistently using the iterative technique of Section 4. The result is given in Figures 6 and 7, where we have adopted standard notation (Fetter and Walecka, 1971) for the propagators $G'^{(0,2)}$ and $G'^{(2,0)}$.

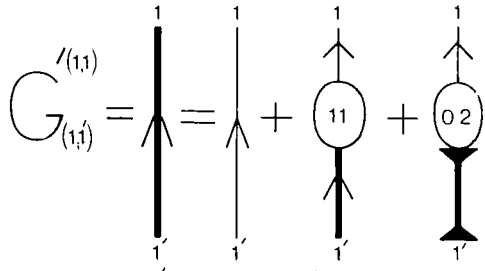


(38)

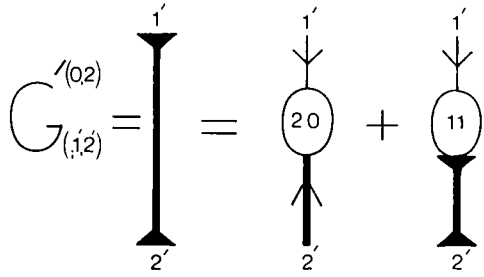


(39)

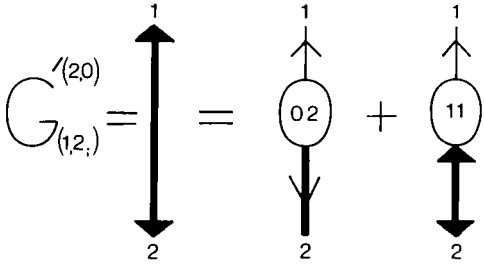
Fig. 5



(40)

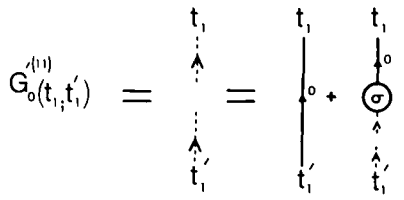


(41)



(42)

Fig. 6



(43)

Fig. 7

Note that apart from the argument immediately preceding (35), equation (2) and its analogs for higher-order condensate Green functions has not yet been employed. Thus, equations (40)–(43) are formally correct even when equation (2) is invalid. However, the self-energies are not decoupled and the equations, when taken separately, are not well defined. To see this, note that the diagram of Figure 8 involves a two particle $\mathbf{k} = 0$ Green function and unless a suitable form for $G_0^{m,m}$ is assumed, for example,

$$i^n G_0^{(m,m)}(t_1 \ldots t_m; t_1' \ldots t_m') = (n_0)^m \exp[-i\mu(t_1 + \ldots + t_m - t_1' \ldots - t_m')] \tag{44}$$

the self-energies cannot be considered as separate entities. For this reason, equation (44) might be termed a "decoupling approximation."

Introducing this approximation, equations (40)–(42) yield the well-known equations for the noncondensate Green function of a condensed Bose system (Beliaev, 1958) and may be combined to yield the following equation for the Fourier transform of $G'^{(1,1)}$:

$$G'^{(1,1)}(\mathbf{k}, \omega + \mu) \equiv G'(k + \mu) = \frac{g'^{-1}(-k+\mu) - \Sigma_{11}(-k+\mu)}{\{[g'^{-1}(k+\mu) - \Sigma_{11}(k+\mu)][g'^{-1}(-k+\mu) - \Sigma_{11}(-k+\mu)] - \Sigma_{02}(k+\mu)\Sigma_{20}(k+\mu)\}} \tag{45}$$

Equation (43) is quite new however and arises because the hierarchy of equations evaluates the whole Green function $G^{(1,1)}$ rather than simply

Fig. 8

$G'^{(1,1)}$. In view of the assumed form for $G_0^{(1,1)}$ given in equation (2b), it might appear that (43) is redundant. However, as shown in Appendix B, the self-energy σ is related to the self-energies of equations (40)–(42) by

$$\sigma(0,\mu) = \Sigma_{11}(0,\mu) - \Sigma_{02}(0,\mu) \tag{46}$$

Furthermore, it is also shown that equation (43) leads to relation

$$\sigma_{(0,\mu)} = \mu \tag{47}$$

Combining these equations yields the well-known Hugenholtz and Pines relation

$$\Sigma_{11}(0,\mu) - \Sigma_{12}(0,\mu) = \mu \tag{48}$$

which may thus be regarded as a self-consistency condition for the theory.

This demonstrates how the Hugenholtz and Pines theory may be obtained by combining a Beliaev type of approach with the hierarchy of equations. Significantly, the self-consistent expression for the $\mathbf{k} = 0$ Green function does not contain inconsistencies and instead provides a means of obtaining self-consistency conditions on the theory.

This is a new development and opens up fresh avenues of analysis in the boson problem. As an example, one might consider alternative *ab initio* assumptions concerning the form of the m particle $\mathbf{k} = 0$ Green functions. From the above analysis, it is clear that an approximation which decouples the self-energies is desirable and given such an approximation, equation (43) may be employed to yield self-consistency conditions on the theory. An extension of the Beliaev, Hugenholtz, and Pines theory has recently been proposed along the above lines (Hagston and Lambert, 1980) and has led to useful predictions in the long-wavelength region of the excitation spectrum of a Bose fluid.

6. CONCLUSION

The aim of the present paper has been to introduce a new approach to perturbation theory based on a hierarchy of integral equations and its solutions. We have sought to illustrate the new techniques involved by deriving some well-known results of perturbation theory. The formalism is extremely general and is applicable to any quantum system which possesses a Hamiltonian. For convenience, we have restricted the analysis to the nonrelativistic case of a many-particle system interacting via an instantaneous two-body potential.

One of the most interesting features of the above formalism is the transparancy of the link between the hierarchy of equations and diagrammatic analysis. This is extremely useful when one wishes to relate a self-consistent approach to the summation of a given subset of diagrams. For example the Hartree–Fock approximation may be obtained, without recourse to perturbation theory, by employing the factorization

$$G^{(2,2)}(1,2;1',2') = G(1;2')G(2;1') + SG(1;1')G(2;2') \qquad (49)$$

Substitution of this equation into the last term on the right-hand side of equation (30) leads to the desired result. Furthermore, if one includes in the factorization (49), an anomolous term of the form $G^{(2,0)}(1,2;)G^{(0,2)}(;1',2')$, one obtains a generalization of Gorkov's approach to the theory of superconductivity (Gorkov, 1958). Since the present analysis is independent of statistics, this feature may have interesting applications within the pairing theories of superfluidity (Evans and Harris, 1978; ter Haar, 1977).

In Section 4, the link with diagrammatic analysis was exploited to yield a well-known perturbation series, without introducing the adiabatic hypothesis. Disconnected diagrams were avoided and Dyson's equations appeared naturally at each stage of the iterative technique employed. The terms of the series are functionals of $F(\mathbf{k})$. This quantity may be determined by examining the limit $H_1 = 0$ or by the minimization principle of equation (34). The latter is expected to be useful in situations where $|\Psi_N\rangle$ is not the adiabatic transform of $|\Phi_N\rangle$. The analysis of Section 4 is applicable to both a finite density Fermi system and a zero density Bose system.

An important consequence of avoiding the use of adiabatic switching is that the Green functions appearing on the right-hand side of equations (28) are defined with respect to the Heisenberg picture of H and not H^{ϵ}. Thus, these functions are exact propagators of the interacting system and in situations where physical arguments are employed to yield expressions for the Green functions, the hierarchy of equations may be used to obtain self-consistency conditions on the resulting theory. An example of this is given in Section 5, in which the self-consistent equation for the condensate Green function [equation (43)] yields the self-consistency condition (47). We emphasize that this technique leads to inconsistencies when coupled within the conventional approach to perturbation theory and hence the present formulation allows a new self-consistent approach to the boson problem. Indeed, the analysis of Section 5 has already been successfully employed (Hagston and Lambert, 1980) to yield a self-consistent extension of the Beliaev, Hugenholtz, and Pines formalism.

Finally, we remark that the present ground state formalism is readily extended to finite temperatures (Lambert, 1981) by employing a finite temperature hierarchy of equations analogous to equations (28). In addition

(Lambert and Hagston, 1981; Hagston and Lambert, 1981) the above formalism has been employed to yield a self-consistent perturbation theory of many-particle systems based on a renormalization of the interparticle potential.

APPENDIX A. DERIVATION OF EQUATION (15)

In the present section, an arbitrary Heisenberg (interaction) picture operator will be represented by $B_H(t_i)(B(t_i))$. To simplify notation, we use t_i to label the operator as well as the "time." Equation (15) is derived by analyzing the first term on the left-hand side and to this end we consider the following time-ordered product of operators:

$$e^{-\varepsilon|t|}e^{i\omega_k t}T\{a_{Hk}(t)B_H(t_1)\ldots B_H(t_l)\} \tag{A1}$$

It is convenient to impose the condition

$$t_1 > t_2 > \ldots > t_{j-1} > t > t_j > \ldots > t_l \tag{A2}$$

which allows us to write

$$\begin{aligned} &e^{-\varepsilon|t|}e^{i\omega_k t}T\{a_{Hk}(t)B_H(t_1)\ldots B_H(t_l)\} \\ &\quad = e^{-\varepsilon|t|}e^{i\omega_k t}S^{j-1}B_H(t_1)\ldots B_H(t_{j-1})a_{Hk}(t)B_H(t_j)\ldots B_H(t_l) \\ &\quad = S^{j-1}U(0,t_1)B(t_1)U(t_1,t_2)\ldots B(t_{j-1})U(t_{j-1},t)a_k e^{-\varepsilon|t|} \\ &\qquad \times U(t,t_j)B(t_j)U(t_j,t_{j+1})B(t_{j+1}) \\ &\qquad \ldots U(t_{l-1},t_l)B(t_l)U(t_l,t'')U(t'',0) \end{aligned} \tag{A3}$$

where the last line follows from the relations

$$B_H(t) = U(0,t)B(t)U(t,0) \tag{A4}$$

$$a_{Ik}(t) = e^{-i\omega_k t}a_k \tag{A5}$$

and the fact that for arbitrary t'',

$$U(t_l,0) = U(t_l,t'')U(t'',0) \tag{A6}$$

We analyze the right-hand side of equation (A3) by commuting a_k to the right until it lies between the operators $U(t_l,t'')$ and $U(t'',0)$. To facilitate this exercise, we note that from the definition, it follows that

$$\frac{\partial}{\partial t'}\left[e^{-\varepsilon|t'|}U(t,t')a_k U(t',t_0)\right] = -iU(t,t')X(t')U(t',t_0) \tag{A7}$$

where

$$X(t') = \{[a_{\mathbf{k}}, H_1(t')] - i\varepsilon \operatorname{sgn}(t') a_{\mathbf{k}}\} e^{-\varepsilon|t'|} \tag{A8}$$

Integrating equation (A7) and noting the boundary condition, yields

$$a_{\mathbf{k}} e^{-\varepsilon|t|} U(t, t_0) = U(t, t_0) a_{\mathbf{k}} e^{-\varepsilon|t_0|} - i \int_{t_0}^{t} dt' U(t, t') X(t') U(t', t_0) \tag{A9}$$

With the help of this equation, we now proceed to commute $a_{\mathbf{k}}$ in equation (A3), to the right. This yields

$$\begin{aligned}
& e^{-\varepsilon|t|} e^{i\omega_{\mathbf{k}} t} T\{a_{H\mathbf{k}}(t) B_H(t_1) \ldots B_H(t_l)\} \\
& \quad = S^{j-1} U(0, t_1) B(t_1) U(t_1, t_2) \ldots B(t_{j-1}) U(t_{j-1}, t) \\
& \quad \Bigg\{ \left[-i \int_{t_j}^{t} dt' U(t, t') X(t') U(t', t_j) \right] B(t_j) U(t_j, t_{j+1}) \ldots \\
& \qquad \times B(t_l) U(t_l, t'') U(t'', 0) + U(t, t_j)[a_{\mathbf{k}}, B(t_j)]_{-S} \\
& \qquad \times e^{-\varepsilon|t_j|} U(t_j, t_{j+1}) B(t_{j+1}) \ldots B(t_l) U(t_l, t'') U(t'', 0) \\
& \qquad + S U(t, t_j) B(t_j) \left[-i \int_{t_{j+1}}^{t_j} dt' U(t_j, t') X(t') U(t', t_{j+1}) \right] \\
& \qquad \times B(t_{j+1}) \ldots B(t_l) U(t_l, t'') U(t'', 0) \\
& \qquad + S U(t, t_j) B(t_j) U(t_j, t_{j+1}) [a_{\mathbf{k}}, B(t_{j+1})]_{-S} e^{-\varepsilon|t_{j+1}|} \ldots \\
& \qquad \times B(t_l) U(t_l, t'') U(t'', 0) \\
& \qquad \qquad \vdots \\
& \qquad + S^{l-j} U(t, t_j) B(t_j) U(t_j, t_{j+1}) B(t_{j+1}) \ldots \\
& \qquad \times [a_{\mathbf{k}}, B(t_l)]_{-S} e^{-\varepsilon|t_l|} U(t_l, t'') U(t'', 0) \\
& \qquad + S^{l-j+1} U(t, t_j) B(t_j) U(t_j, t_{j+1}) B(t_{j+1}) \ldots \\
& \qquad \times B(t_l) \left[-i \int_{t''}^{t_l} dt' U(t_l, t') X(t') U(t', t'') \right] U(t'', 0) \\
& \qquad + S^{l-j+1} U(t, t_j) B(t_j) U(t_j, t_{j+1}) B(t_{j+1}) \ldots \\
& \qquad \times B(t_l) U(t_l, t'') a_{\mathbf{k}} U(t'', 0) e^{-\varepsilon|t''|} \Bigg\}
\end{aligned} \tag{A10}$$

Noting that

$$U(0,t'')a_{\mathbf{k}}U(t'',0) = e^{i\omega_{\mathbf{k}}t''}a_{H\mathbf{k}}(t'') \tag{A11}$$

and

$$\int_{t_{i+1}}^{t_i} dt' U(t_i,t')\,X(t')U(t',t_{i+1}) = U(t_i,0)\left[\int_{t_{i+1}}^{t_i} dt' e^{i\omega_{\mathbf{k}}t'}X_H(t')\right]U(0,t_{i+1}) \tag{A12}$$

where

$$e^{i\omega_{\mathbf{k}}t'}X_H(t') = U(0,t')\,X(t')U(t',0) \tag{A13}$$

and choosing t'' to satisfy $t_l > t''$, equation (A10) becomes

$$\begin{aligned} e^{-\varepsilon|t|}e^{i\omega_{\mathbf{k}}t}T\{a_{H\mathbf{k}}(t)B_H(t_1)\ldots B_H(t_l)\} &= \sum_{r=1}^{l} S^{r-1}e^{-\varepsilon|t_r|}e^{i\omega_{\mathbf{k}}t_r}B_H(t_1)\ldots B_H(t_{r-1}) \\ &\quad\times\left[a_{H\mathbf{k}}(t_r), B_H(t_r)\right]_{-S}B_H(t_{r+1})\ldots B_H(t_l)\theta(t-t_r) \\ &\quad - i\int_{t''}^{t} dt' T\{X_H(t')B_H(t_1)\ldots B_H(t_l)\}e^{i\omega_{\mathbf{k}}t'} \\ &\quad + S^l e^{-\varepsilon|t''|}e^{i\omega_{\mathbf{k}}t''}B_H(t_1)\ldots B_H(t_l)a_{H\mathbf{k}}(t'') \end{aligned} \tag{A14}$$

In view of the condition (A2), and the remarks above equation (17) of the text, this may be written

$$\begin{aligned} e^{-\varepsilon|t|}e^{i\omega_{\mathbf{k}}t}T\{a_{H\mathbf{k}}(t)B_H(t_1)\ldots B_H(t_l)\} &= \sum_{r=1}^{l} S^{r-1}e^{-\varepsilon|t_r|}e^{i\omega_{\mathbf{k}}t_r}T \\ &\quad\times\{B_H(t_1)\ldots B_H(t_{r-1})\left[a_{H\mathbf{k}}(t_r), B_H(t_r)\right]_{-S} \\ &\qquad\times B_H(t_{r+1})\ldots B_H(t_l)\}\theta(t-t_r) \\ &\quad - i\int_{t''}^{t} dt' T\{X_H(t')B_H(t_1)\ldots B_H(t_l)\}e^{i\omega_{\mathbf{k}}t'} \\ &\qquad + e^{-\varepsilon|t''|}e^{i\omega_{\mathbf{k}}t''}T\{a_{H\mathbf{k}}(t'')B_H(t_1)\ldots B_H(t_l)\} \end{aligned} \tag{A15}$$

This equation is only valid for $t_l > t''$. To remove this restriction, consider repeating the analysis for some other value of t. Denoting this value of t by T and subtracting the result from equation (A15) yields

$$
\begin{aligned}
& e^{-\epsilon|t|}e^{i\omega_{\mathbf{k}}t}T\{a_{H\mathbf{k}}(t)B_H(t_1)\ldots B_H(t_l)\} \\
& \quad - e^{-\epsilon|T|}e^{i\omega_{\mathbf{k}}T}T\{a_{H\mathbf{k}}(t)B_H(t_1)\ldots B_H(t_l)\} \\
& = \sum_{r=1}^{l} S^{r-1}e^{-\epsilon|t_r|}e^{i\omega_{\mathbf{k}}t_r}T \\
& \quad \times\{B_H(t_1)\ldots B_H(t_{r-1})[a_{H\mathbf{k}}(t_r), B_H(t_r)]_{-S}B_H(t_r+1)\ldots B_H(t_l)\} \\
& \quad \times\{\theta(t-t_r)-\theta(T-t_r)\}-i\int_T^t dt'T\{X_H(t')B_H(t_1)\ldots B_H(t_l)\}e^{i\omega_{\mathbf{k}}t'}
\end{aligned}
\tag{A16}
$$

This equation is valid for arbitrary t and T. Furthermore, it is valid for arbitrary $\{t_i\}$. To see this, substitute equation (17) into the first term on the right-hand side. Each term in the resulting equation contains l operators $B(t_i)$. Hence for arbitrary $\{t_i\}$ the operators $B(t_i)\ldots B(t_l)$ can be simultaneously reordered on both sides and any factors S^P resulting from P permutations cancel exactly.

The derivation of equation (15) is completed after replacing $B_H(t_i)\ldots B_H(t_l)$ by $\psi_H(2)\ldots\psi_H(m)\psi_H^\dagger(1')\ldots\psi_H^\dagger(n')$ and noting that

$$[a_{\mathbf{k}}, \psi_{(j)}]_{-S} = 0.$$

APPENDIX B: DERIVATION OF EQUATIONS (46) AND (47)

In the thermodynamic limit, the first term on the right-hand side of equation (43) vanishes and a combination of this equation with (44) yields

$$n_0 e^{-i\mu(t_1-t_1')} = n_0 e^{-i\mu(t_1-t_1')}\frac{\sigma(0,\mu)}{\mu} \tag{B1}$$

where $\sigma(\mathbf{k},\omega)$ is the Fourier transform of the self-energy σ. Hence, from (B1)

$$\sigma(0,\mu) = \mu \tag{B2}$$

We proceed to demonstrate that

$$\sigma(0,\mu) = \Sigma_{11}(0,\mu) - \Sigma_{02}(0,\mu) \tag{B3}$$

Our analysis is essentially based on a trivial modification of the arguments employed by Abrikosov et al. (1963) to derive equation (47).

The average of $U_\varepsilon(t_0, -t_0)$ with respect to the vacuum of the operators ψ' and $\psi'^\dagger$ is given by (Beliaev, 1958)

$$\bar{U}(t_0, -t_0) = T[e^{v\gamma}] \tag{B4}$$

where γ is a functional of a_0 and $a_0^\dagger$. γ is a sum of vacuum loops (Beliaev, 1958) and in general, each contribution to γ contains m pairs of interaction picture operators $a_0(t_1)\ldots a_0(t_m)a_0^\dagger(t_1')\ldots a_0^\dagger(t_m')$. Let $\bar{\gamma}$ be the function obtained by replacing such strings of operators in γ by (36). (In this way, we replace operators $a_0(t_i)[a_0^\dagger(t_i')]$ by dotted lines entering (leaving) the points $t_i[t_i']$.) Then $\bar{\gamma}$ will be a c-number function which may be written

$$\bar{\gamma} = \sum_m \bar{\gamma}_m \tag{B5}$$

where $\bar{\gamma}_m$ is the sum of all vacuum loops containing m pairs of dotted lines. A careful consideration of the analysis which led to equations (40)–(43) reveals the following:

(i) $\sigma(0, \mu)$ is obtained by first removing a single arbitrarily chosen ingoing dotted line from $\bar{\gamma}$ and then removing in all possible ways an outgoing dotted line. Thus

$$\sigma(0, \mu) = \sum_m \frac{m}{n_0} \bar{\gamma}_m \tag{B6}$$

(ii) $\Sigma_{11}(0, \mu)$ is obtained by removing an ingoing and outgoing dotted line from $\bar{\gamma}$ in all possible ways:

$$\Sigma_{11}(0, \mu) = \sum_m \frac{m^2}{n_0} \bar{\gamma}_m \tag{B7}$$

(iii) Finally, $\Sigma_{02}(0, \mu)$ is obtained by removing a pair of outgoing dotted lines from $\bar{\gamma}$ in all possible ways:

$$\Sigma_{02}(0, \mu) = \sum_m \frac{m(m-1)}{n_0} \bar{\gamma}_m \tag{B8}$$

A combination of equations (B2), (B6) to (B8) yields the desired result.

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